

stants only up to and including third-order are used. This would imply that a shock wave in this direction is unstable and spreads as it travels. With the addition of the fourth-order constant the curve has a minimum at a value of $u = 0.032$ mm/ μ sec. The minimum sound speed occurs at a compression ratio $V/V_0 = 0.9830$ and a stress $\sigma_x = 15.4$ kb. Its value there is 5.6889×10^5 cm/sec. The critical shock strength above which a single shock is stable occurs at $u = 0.063$ mm/ μ sec, $V/V_0 = 0.9890$, and $\sigma_x = 9.6$ kb. The lack of a unique shock speed below the critical value probably does not influence the operation of quartz shock transducers appreciably, because the variation in speed is small ($<1\%$).

The U_s - U_p plot for Z-cut crystals is nearly a straight line; however, it is easily shown that a straight line does not accurately fit the slope at zero particle velocity. Thus, the straight line relation often assumed in shock studies is only an approximation for quartz shocked in either the X or the Z direction.

It is also easily shown that the Murnaghan form of the equation of state, when fitted to the correct slope and curvature of the σ - V curve (utilizing second- and third-order constants), does not accurately fit the higher-pressure data and is therefore an approximation only.

These statements can be illustrated by examining the derivatives of each function. Expanding the relation for σ in (11) in terms of γ yields

$$-\sigma = c_{11}\gamma \left[1 - \frac{1}{2} \left(3 + \frac{c_{111}}{c_{11}} \right) \gamma + \frac{1}{6} \left(3 + 6 \frac{c_{111}}{c_{11}} + \frac{c_{1111}}{c_{11}} \right) \gamma^2 + \dots \right] \quad (12)$$

A linear relation between shock and particle velocity of the form

$$V_s = a + bu_p$$

can be written, by means of the Rankine-Hugoniot jump conditions, as

$$-\sigma = \rho_0 [a^2 \gamma / (1 - b\gamma)^2]$$

where ρ_0 is initial density, and the relation can be expanded to give

$$-\sigma = \rho_0 a^2 \gamma [1 + 2b\gamma + 3b^2 \gamma^2 + \dots] \quad (13)$$

Finally, the one-dimensional strain analog of the Murnaghan equation

$$-\sigma_1 = (A/B) [(V_0/V)^B - 1]$$

can be expanded to give

$$-\sigma = A\gamma [1 + \frac{1}{2}(B+1)\gamma + \frac{1}{6}(B+1)(B+2)\gamma^2 + \dots] \quad (14)$$

Equating the derivatives up to second-order, we have

$$c_{11} = \rho_0 a^2 = A \quad (1\text{st order})$$

$$-\left(3 + \frac{c_{111}}{c_{11}} \right) = 4b = (B+1) \quad (2\text{nd order})$$

Evaluating the parameters A , b , and B from these equations, we have

X cut

$$A = \rho_0 a^2 = c_{11} = 8.757 \times 10^{11}$$

$$b = -0.15 \quad B = -1.6$$

Z cut

$$A = \rho_0 a^2 = c_{33} = 10.575 \times 10^{11}$$

$$b = 1.177 \quad B = 3.71$$

With these values all three expressions have the same slope and curvature at zero stress. The predicted stresses for various compressions are shown in Table 3.

That the closed form expressions are approximate is hardly surprising inasmuch as they are both empirical with no known physical basis. They are of value because they both are two-parameter functions that have physically reasonable shapes and they are, therefore, convenient for interpolation and extrapolation when experimental information is lacking.

The close agreement of the Murnaghan and linear U_s - u_p fits is somewhat curious. It presumably results from the fact that the third- and higher-order coefficients in the expansions of these formulas are relatively small; they must agree exactly, of course, to terms of second order. Direct calculation shows that the coefficient of the third-order term in (12) is, for each crystal orientation, a factor of 5 or more greater than the corresponding term in (13) and (14).

TABLE 3. Stress in Kilobars

V/V ₀	X			Z		
	Murnaghan	Linear U _s -U _p	Finite Strain	Murnaghan	Linear U _s -U _p	Finite Strain
0.99	8.7	8.8	8.7	10.7	10.8	10.8
0.98	17.4	17.5	17.5	21.8	22.2	22.3
0.97	26.1	26.3	26.5	32.0	34.0	34.5
0.96	34.7	35.0	35.9	47.1	46.5	47.6
0.95	43.1	43.7	45.7	60.3	59.7	61.6
0.94	51.7	52.5	56.1	73.9	73.5	76.5
0.93	59.9	61.3	67.1	87.9	88.0	92.6
0.92	68.6	70.0	78.9	103.4	103.0	109.7
0.91	77.0	78.8	91.4	119.6	119.0	127.9
0.90	85.2	87.6	104.7	136.4	135.8	147.3

Knopoff's [1963] suggestion that because of the arbitrariness in the definition of strain alternative definitions may prove more suitable for representing constitutive relations would seem to be worthy of further consideration. Some guidance from physical reasoning is necessary, however, to provide any degree of generality to a given definition.

Error analysis. The precision of the fourth-order constants is determined by the precision of the shock data and by the precision of the second- and third-order constants. Of these the error in the shock data has the largest effect.

Differentiation of the stress-strain relation shows that the error in c_{1111} due to an error in c_{11} is

$$\frac{dc_{1111}}{c_{1111}} = \left(-\frac{6c_{11}}{c_{1111}N_1^2} \right) \frac{dc_{11}}{c_{11}}$$

and for strains $N_1 = 6 \times 10^{-2}$ the coefficient is approximately -10 . Thus, a 0.1% error in c_{11} results in a 1% error in c_{1111} . Similarly, an error in the third-order constant c_{111} is equivalent to an error in c_{1111} of

$$\frac{dc_{1111}}{c_{1111}} = \left(-\frac{3c_{111}}{c_{1111}N_1} \right) \frac{dc_{111}}{c_{111}}$$

and the coefficient for strains of approximately 6×10^{-2} is 0.5. For Z-cut crystals the corresponding coefficients are approximately 5 and 2.

Taking the precision of the second- and third-order constants to be approximately ± 0.1 and $\pm 5\%$, respectively [McSkimin *et al.*, 1965; Thurston *et al.*, 1966], the error in the fourth-

order constants from these sources is approximately ± 3 and $\pm 10\%$ for X- and Z-cut crystals, respectively.

The error due to inaccuracies in the shock data is evaluated as follows. From the jump conditions (equation 3) we have

$$1 - (V/V_0) = u/U$$

and

$$\sigma = \rho_0 U u$$

Hence,

$$-\frac{\delta V}{V_0} = \frac{u}{U} \left(\frac{\delta u}{u} - \frac{\delta U}{U} \right)$$

and

$$\delta\sigma/\sigma = \delta u/u + \delta U/U$$

We are interested, however, not in the total error in the measured state but in the error in the Hugoniot stress-strain curve. That is, we wish to know the error in stress at a given specific volume.

If the true slope of the Hugoniot is $d\sigma/dV$, the quantity of interest is

$$\frac{\Delta\sigma}{\sigma} = \left(\frac{d\sigma}{dV} - \frac{\delta\sigma}{\delta V} \right) \frac{\delta V}{\sigma}$$

This can be reduced to

$$\frac{\Delta\sigma}{\sigma} = \frac{\delta U}{U} \left[\frac{d\sigma/dV}{\sigma/(V_0 - V)} - 1 \right] - \frac{\delta u}{u} \left[\frac{d\sigma/dV}{\sigma/(V_0 - V)} + 1 \right]$$